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On Certain Minimal Conditions for Infinite Groups

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1. INTRODUCTION

1.1. In a number of papers, Černikov has studied groups “many” of whose subgroups have some given property \mathcal{P} . Two of Černikov’s interpretations of the requirement that “many subgroups have \mathcal{P} ” have been that “all infinite subgroups have \mathcal{P} ”, and, more generally, that “the set of subgroups not having \mathcal{P} satisfies the minimal condition”. Conditions \mathcal{P} which have been investigated in this context have been absolute properties (such as commutativity [7] and non-commutativity [6, 9 and many other papers by various authors]), embedding properties (such as the properties of being a normal subgroup [8, 10] and of being an ascendant subgroup [8]), or sometimes combinations of both [11]. The results have included structure theorems, and sometimes assertions that certain groups “many” but not all of whose subgroups have \mathcal{P} are Černikov groups, that is, they are Abelian by finite groups satisfying the minimal condition for subgroups. For a fuller description of some of these results and other related ones, we refer the reader to Černikov’s survey paper [9].

Our object here is to take this programme of research a stage further by studying groups satisfying conditions somewhat weaker than many of those hitherto studied in this context. We shall be primarily concerned with groups satisfying the minimal condition for subgroups not having \mathcal{P} , where \mathcal{P} is taken to be the property of being either a serial subgroup or locally nilpotent. Our main theorem implies that any group satisfying this condition, and satisfying an additional condition (weaker than local finiteness and local solubility) whose significance

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will be explained below, either is a locally-nilpotent by finite-cyclic group all of whose subgroups have \mathcal{P} , or is a Černikov group. Of course, the Šunkov–Kegel–Wehrfritz theorem, that every locally finite group satisfying the minimal condition for subgroups is a Černikov group, is one consequence of this result. We shall also prove that, for various properties \mathcal{P}' stronger than \mathcal{P} , any group satisfying the minimal condition for subgroups not having \mathcal{P}' , and again satisfying an additional condition, either has no subgroups not having \mathcal{P}' or is a Černikov group. Among such properties \mathcal{P}' are the property of being Abelian or a serial subgroup and the property of being Abelian or a normal subgroup.

1.2. Before stating our main theorem precisely, we must discuss the “additional condition” alluded to above, and also right Engel sets and a certain class \mathcal{A} of finite groups.

An attempt to prove structure theorems for all groups satisfying even a very strong condition on subgroups is likely to founder on the question of the existence of so-called “Tarski groups”—infinite groups all of whose proper non-trivial subgroups have the same prime order. This and similar difficulties have usually been avoided by the imposition of some extra condition. For example, when discussing groups all of whose infinite subgroups are Abelian or normal in [11], Černikov considers only *locally graded* groups—groups each of whose non-trivial finitely generated subgroups has a proper subgroup of finite index. Our work is further complicated by the difficulty of describing satisfactorily the finitely generated groups all of whose subgroups are serial subgroups. Non-nilpotent such groups were constructed in [36]: the examples are residually finite and so locally graded, and each of their finite images is nilpotent. We shall work with the class \mathfrak{B} of groups each of whose finitely generated subgroups either is nilpotent or has a non-nilpotent finite quotient group. It is obvious that all locally finite groups are \mathfrak{B} -groups and that all \mathfrak{B} -groups are locally graded. It follows from a theorem of Robinson [23] that \mathfrak{B} contains all locally hyper-(Abelian or finite) groups, and, in particular, all locally soluble groups. Finally, \mathfrak{B} contains all linear groups (see Wehrfritz [33]).

For any group G , we write $R(G)$ for the set of right Engel elements of G . Thus $R(G)$ is the set of $x \in G$ such that, for each $y \in G$, the commutator

$$[x, \underbrace{y, \dots, y}_n]$$

is trivial for large enough n . It is not known whether or not $R(G)$ is in general a subgroup; however it is shown in Lemma 6 below that $R(G)$ will be a locally nilpotent normal subgroup of G if G is a \mathfrak{B} -group. We shall show that \mathfrak{B} -groups satisfying the minimal condition for non-serial non-locally-nilpotent subgroups are locally (finite by nilpotent) groups, and in such groups $R(G)$ behaves very much like the hypercentre of a finite group. Writing X ser Y to mean that X is a serial subgroup of Y , we shall prove

PROPOSITION 1. *Let G be a locally Noetherian group.*

- (a) *If $H \leq G$, then H is locally nilpotent if and only if $R(G)H$ is locally nilpotent.*
- (b) *If $H \leq G$, then $H \text{ ser } G$ if and only if $R(G)H \text{ ser } G$.*
- (c) *If also G is locally (finite by nilpotent), then $R(G)$ is the largest normal subgroup of G all of whose G -chief factors are central factors of G .*

Of course, locally (finite by nilpotent) groups are locally Noetherian.

The first two assertions of Proposition 1 show a similarity between serial subgroups and locally nilpotent subgroups of a locally Noetherian group. Moreover it is well known that all subgroups of locally nilpotent groups are serial subgroups (cf. Kuroš [21, p. 221]) and it is extremely easy to prove that every \mathfrak{B} -group all of whose subgroups are serial subgroups is locally nilpotent. These relationships between seriality and local nilpotence provided part of the motivation for combining the properties.

The class Δ referred to above is the class of finite centreless groups each of whose subgroups is a subnormal subgroup or is nilpotent. It is rather simple to classify Δ -groups completely, and we will describe their structure in Proposition 2 below: nontrivial Δ -groups are cyclic extensions of Abelian minimal normal subgroups, and each of their subgroups is either a subnormal subgroup (of defect at most two) or is cyclic. By definition of the class Δ , any finite group each of whose subgroups is a subnormal subgroup or is nilpotent is an extension of its hypercentre by a Δ -group.

We may now state our main theorem.

THEOREM A. *Let G be a \mathfrak{B} -group. The following conditions are equivalent:*

- (a) *G satisfies the minimal condition for non-serial non-locally-nilpotent subgroups,*
- (b) *either G is a Černikov group, or G is locally (finite by nilpotent) and $R(G)$ is a locally nilpotent normal subgroup such that $G/R(G) \in \Delta$,*
- (c) *either G is a Černikov group, or each subgroup of G is a serial subgroup or is locally nilpotent.*

In particular, if G satisfies (a) and is not a Černikov group, then G is locally nilpotent by finite cyclic.

Some of the significance of $R(G)$ in assertion (b) here may be seen from Proposition 1 above; and indeed the implication (b) \Rightarrow (c) and the fact that a group satisfying (b) is either a Černikov group or is locally nilpotent by finite cyclic follow immediately from this Proposition and the remarks about the class Δ following it. Because it is quite clear that (c) implies (a), the main content of Theorem A is the implication (a) \Rightarrow (b).

We list some immediate consequences of Theorem A in

COROLLARY A1. *Let G be a \mathfrak{B} -group satisfying the minimal condition for non-serial non-locally-nilpotent subgroups.*

(a) *If G either is uncountable or has an element of infinite order, then each subgroup of G is a serial subgroup or is locally nilpotent.*

(b) *If G is torsion-free, then G is locally nilpotent.*

Assertion (a) here, which is similar to results proved by Černikov for groups satisfying other conditions on their subgroups, follows because Černikov groups are countable torsion groups. Assertion (b) comes from the fact that, according to Theorem A, the group G will be locally (finite by nilpotent).

Another corollary, which we have already mentioned, is

COROLLARY A2. (*Šunkov [31], Kegel and Wehrfritz [19]*). *All locally finite groups satisfying the minimal condition for subgroups are Černikov groups.*

This follows because, by a theorem of Černikov, all locally nilpotent groups satisfying the minimal condition are Černikov groups (cf. Černikov [6], or [20, Theorem 1, E.6]). While Corollary A2 is not used in the proof of Theorem A, our arguments do depend heavily on the techniques developed by Šunkov, Kegel and Wehrfritz for the study of locally finite simple groups satisfying the minimal condition, and therefore also on the deep results from finite group theory which lie behind them.

1.3. A simple way to obtain stronger chain conditions than the minimal condition for non-serial non-locally-nilpotent subgroups is to replace independently the embedding property of seriality and the absolute property of local nilpotence by stronger properties. We have chosen to strengthen seriality to normality and to the property of being the trivial subgroup (the latter to yield minimal conditions for subgroups having absolute properties), and to strengthen local nilpotence principally to commutativity and to the property of being trivial (to yield minimal conditions for subgroups having embedding properties). Independent substitutions of these properties yield nine minimal conditions. Two of these nine conditions, the minimal condition for non-serial non-locally-nilpotent subgroups and the minimal condition for subgroups, have already been discussed; the other seven are

- (i) *the minimal condition for non-serial non-Abelian subgroups,*
- (ii) *the minimal condition for non-serial subgroups,*
- (iii) *the minimal condition for non-normal non-locally-nilpotent subgroups,*
- (iv) *the minimal condition for non-normal non-Abelian subgroups,*
- (v) *the minimal condition for non-normal subgroups,*
- (vi) *the minimal condition for non-locally-nilpotent subgroups, and*
- (vii) *the minimal condition for non-Abelian subgroups.*

If we let \mathfrak{X} stand for an arbitrary subgroup-closed class of locally nilpotent groups which satisfies one of the conditions (a) \mathfrak{X} is closed under normal products of two subgroups, (b) \mathfrak{X} is closed under quotients and nilpotent subdirect products, or (c) \mathfrak{X} contains only periodic groups, then each of the conditions (iii)–(vii) is either of the form

(viii) *the minimal condition for non-normal non-locally- \mathfrak{X} subgroups,*
or of the form

(ix) *the minimal condition for non-locally- \mathfrak{X} subgroups.*

For all of these minimal conditions (and for yet more which might have been interpolated), there are results of the form “if G is a \mathfrak{B} -group satisfying the minimal condition for subgroups not having \mathcal{P} , then either G is a Černikov group (and so satisfies the minimal condition for subgroups) or every subgroup of G has \mathcal{P} .” These nine results we refer to collectively as Theorem B and individually as B(i), ..., B(ix). We note that each assertion of Theorem B has a trivial converse.

Conditions (viii) and (ix) have been included partly to allow economy of proof—four assertions have to be proved for Theorem B instead of seven—but more to illustrate the profusion of results of the form “if G is a \mathfrak{B} -group satisfying the minimal condition for subgroups not having \mathcal{P} , then either G is a Černikov group or every subgroup of G has \mathcal{P} .” Possible choices for \mathfrak{X} in B(viii) and B(ix) are the class of nilpotent groups of class at most c , for each integer c , and, more generally, the intersection of any variety of groups with the class of locally nilpotent groups.

Assertions B(i) and B(ii) will follow from more general results describing the structure of \mathfrak{B} -groups satisfying (i) and (ii). Obviously (i) and (ii) are satisfied by Černikov groups, and also by locally nilpotent groups, because all subgroups of locally nilpotent groups are serial subgroups (see Kuroš [21, p. 221]). We shall prove in Theorem C(ii) that *any \mathfrak{B} -group satisfying (ii) either is a Černikov group or is locally nilpotent*; the other \mathfrak{B} -groups satisfying (i) are described in

THEOREM C(i). *Let G be a \mathfrak{B} -group which is neither a Černikov group nor locally nilpotent. The following conditions are equivalent:*

- (a) *G satisfies the minimal condition for non-serial non-Abelian subgroups,*
- (b) *the central quotient group of G is in Δ ,*
- (c) *each subgroup of G either is a serial subgroup of G or is Abelian, and*
- (d) *each subgroup of G either is a subnormal subgroup or is Abelian.*

The implications (d) \Rightarrow (c) and (c) \Rightarrow (a) here are immediate. Furthermore, if G satisfies (b) and has centre Z , and if $H \leq G$, then, from the remarks about Δ -groups in Section 1.2 above, either HZ/Z is subnormal (of defect at most two)

in G/Z or HZ/Z is cyclic; in the first case, because $H \triangleleft HZ$, the subgroup H is subnormal (of defect at most three), while in the second case H is Abelian. Therefore (d) follows from (b), and only the implication (a) \Rightarrow (b) of Theorem C(ii) requires proof.

Groups all of whose subgroups are normal or Abelian, the so-called *metahamiltonian groups*, have been investigated in three papers by Romalis and Sesekin [24, 25 and 26]. It has been shown that every metahamiltonian group is an extension of a group all of whose proper subgroups are Abelian by a metabelian group, and that locally soluble metahamiltonian groups are soluble of derived length (at most) three and have finite derived groups. It is rather easy to see that locally graded groups all of whose proper subgroups are Abelian are metabelian; thus it follows in particular from B(iv) and the results of Romalis and Sesekin that *each locally graded group satisfying the minimal condition for non-normal non-Abelian subgroups either is a Černikov group or has finite metabelian derived group*. One consequence of this is that the requirement in B(iv), B(v) and B(vii) that G be a \mathfrak{B} -group can be relaxed to the requirement that G be locally graded.

Assertion B(v) may be compared with results proved in Černikov [12] concerning groups satisfying the minimal condition for non-normal Abelian subgroups. Both B(v) and B(vii) have been proved by Černikov [10, 13] under the hypothesis that G has a series with finite factors.

Theorems A and B reduce the investigations of \mathfrak{B} -groups all of whose infinite subgroups have \mathcal{P} but not all of whose subgroups have \mathcal{P} , for a number of properties \mathcal{P} , to the study of Černikov groups all of whose infinite subgroups have \mathcal{P} . We have not attempted to study conditions weaker than the minimal condition for non-serial non-locally-nilpotent subgroups. New difficulties seem likely to arise. For example, it is known that the simple groups $\text{PSL}_2(F)$ and $\text{Sz}(F)$, for suitable small infinite locally finite fields F , have all of their infinite subgroups metabelian (see Šafiro [27]). However methods like those used in Section 4 below may be used to show that soluble locally finite groups satisfying the minimal condition for non-metabelian subgroups are either Černikov groups or metabelian, and a characterization of all \mathfrak{B} -groups satisfying the minimal condition for non-metabelian subgroups does not seem out of the question.

1.4. Layout of the paper.

In Section 2 we prove Proposition 1 (on right Engel subgroups), Proposition 2 (on the structure of Δ -groups), together with a number of miscellaneous lemmas for later use. The proof of the implication (a) \Rightarrow (b) of Theorem A is carried out in Section 3. Finally, in Section 4, we prove the outstanding parts of Theorems B and C, namely assertions B(viii) and B(ix), the implication (a) \Rightarrow (b) of Theorem C(i), and Theorem C(ii). The proofs in Section 4 depend only on Section 2 and Theorem A, and not on the arguments used in Section 3 in the proof of Theorem A.

2. PRELIMINARY RESULTS

2.1. In this section we discuss Engel subgroups and prove Proposition 1. We make extensive use of a theorem of Baer [4, p. 257], which asserts that if G is a Noetherian group, then $R(G)$ coincides with the hypercentre of G . We list some immediate consequences of this theorem and the fact that, for any group G and subgroup H , one has $R(H) \geq H \cap R(G)$, in

LEMMA 1. *Let G be a locally Noetherian group.*

- (a) *$R(G)$ is a locally nilpotent characteristic subgroup of G .*
- (b) *$R(G/R(G)) = 1$.*
- (c) *A subgroup S of G satisfies $S \leq R(G)$ if and only if $\langle H, S \rangle$ is locally nilpotent whenever H is a locally nilpotent subgroup of G .*

We note that statement (c) here, with $S = R(G)$, yields statement (a) of Proposition 1. As a step towards the proof of statement (b) of Proposition 1 we prove

LEMMA 2. *If G is a locally Noetherian group and $H \leq G$, then $H \text{ ser } R(G)H$.*

Proof. We show in fact that, if \mathcal{C} is any maximal chain of subgroups from H to $R(G)H$, then, whenever $L, M \in \mathcal{C}$ and L is a maximal subgroup of M , one has $L \triangleleft M$. Of course, such chains exist by Zorn's Lemma, and Lemma 2 follows. Thus suppose that L is a maximal subgroup of M but is not normal in M . We have $R(G) \cap M \leq R(M)$, so that $M = R(M)L$, and, for some $t \in R(M)$, we have $M = \langle L, L^t \rangle$. Because $t \in M$, there is a finitely generated subgroup L_1 of L such that $t \in \langle L_1, L_1^t \rangle$. We write $M_1 = \langle L_1, t \rangle$. Then $M_1 > L_1$, and there is a maximal subgroup L_2 of M_1 containing L_1 . Thus $M_1 = \langle L_2, L_2^t \rangle$ and $t \in R(M) \cap M_1 \leq R(M_1)$. Because M_1 is Noetherian, t lies in the n th term $\zeta_n(M_1)$ of the upper central series of M_1 , for some integer n . It follows that

$$M_1 = L_2^{\zeta_n(M_1)} = L_2 \zeta_{n-1}(M_1).$$

If the integer r is chosen minimal such that $M_1 = L_2 \zeta_r(M_1)$, then $r \geq 1$ and $\zeta_{r-1}(M_1) \leq L_2$. We conclude that

$$[M_1, L_2] \leq L_2 \zeta_{r-1}(M_1) \leq L_2,$$

and that $L_2 \triangleleft M_1$. Because $M_1 = \langle L_2, L_2^t \rangle$, we deduce that $L_2 = M_1$, and this is a contradiction.

We come now to assertion (b) of Proposition 1:

LEMMA 3. *If H is a subgroup of a locally Noetherian group G , then $H \text{ ser } G$ if and only if $R(G)H \text{ ser } G$.*

Proof. It follows immediately from Lemma 2 that, if $R(G)H \text{ ser } G$, then $H \text{ ser } G$. We begin by proving the reverse implication in the special case in which G is Noetherian; then, by Baer's Theorem mentioned above, $R(G)$ coincides with the n th term of the upper central series of G for some integer n , and, by induction, it will suffice to show that if $H \text{ ser } G$, then $ZH \text{ ser } G$, where Z denotes the centre of G . We suppose that this is not so, and replace H by a maximal serial subgroup for which this fails to be true. If $H \triangleleft K \text{ ser } G$ with $H < K$, then $ZK \text{ ser } G$ and $H \triangleleft ZK$, and a contradiction ensues. Thus H must be an intersection of a chain (H_α) of serial subgroups satisfying $H < H_\alpha$. Because intersections of chains of serial subgroups are serial subgroups, we have $\bigcap (ZH_\alpha) \text{ ser } G$. However it is easy to check that $\bigcap H_\alpha \triangleleft \bigcap (ZH_\alpha)$, and again we have a contradiction. Therefore our result certainly holds if G is Noetherian.

We now use a result of Hickin and Phillips [17] to pass to the general case, in which G is locally Noetherian and $H \text{ ser } G$. Suppose that K is a finitely generated subgroup of G , and that $K \leq R(G)(H^K)$. Then $K \leq L(H_1^K)$ for finitely generated subgroups $L \leq R(G)$ and $H_1 \leq H$. Writing $G_1 = \langle H_1, K, L \rangle$, we have

$$H_1 \leq H \cap G_1 \text{ ser } G_1.$$

Because G_1 is finitely generated and so Noetherian, we have

$$R(G_1)(H \cap G_1) \text{ ser } G_1,$$

from the paragraph above, and therefore we have

$$\langle L, H \cap G_1 \rangle \text{ ser } G_1,$$

from Lemma 2 and the observation that $L \leq R(G_1)$. Thus, by Theorem 2 of [17], it follows that $K \leq \langle H \cap G_1, L \rangle$. We conclude that $K \leq R(G)H$, and Theorem 2 of [17] may be used again to show that $R(G)H \text{ ser } G$.

The final assertion of Proposition 1 follows from

LEMMA 4. (a) *If G is locally Noetherian and X/Y is a chief factor of G with $X \leq R(G)$, then X/Y is a central factor of G .*

(b) *If G is locally (finite by nilpotent) and S is a normal subgroup of G such that every chief factor X/Y of G with $X \leq S$ is a central factor of G , then $S \leq R(G)$.*

Proof. (a) Without loss of generality we may suppose $Y = 1$. If X is not a central subgroup of G there are elements $x \in X$ and $g \in G$ with $t = [x, g] \neq 1$; and because X is a minimal normal subgroup we have $X = \langle t \rangle^G$, so that there will be a finitely generated subgroup G_1 of G with $x \in \langle t \rangle^{G_1}$. Setting $D = \langle t, g, G_1 \rangle$, we have

$$t = [x, g] \in [\langle t \rangle^D, D],$$

and therefore $\langle t \rangle^D = [\langle t \rangle^D, D]$. On the other hand,

$$\langle t \rangle^D \leq R(G) \cap D \leq R(D),$$

and $R(D)$ is the hypercentre of the Noetherian group D , so that $[\langle t \rangle^D, D] < \langle t \rangle^D$, a contradiction.

(b) Refining the series $1 \leq S \triangleleft G$ to a chief series of G , we see that S has a series all of whose factors are central factors of G . Thus, for each $x \in G$, the group $K = S\langle x \rangle$ has a series each of whose factors is a central factor of K . If K_1 is a finitely generated subgroup of K , then K_1 also has such a series, \mathcal{S} say, and K_1 also has a finite normal subgroup F such that K_1/F is nilpotent. Because the intersections of the members of \mathcal{S} with F comprise a finite series each of whose factors is a central factor of K_1 , it follows that K_1 is nilpotent. Thus $K = S\langle x \rangle$ is locally nilpotent, and because x here is arbitrary, we conclude (from the definition of $R(G)$) that $S \leq R(G)$, as required.

We note that we have in fact proved more than was needed for two of the three assertions of Proposition 1: Lemma 1(c) and Lemma 4 actually provide characterizations of $R(G)$ in terms of locally nilpotent subgroups and chief factors. We record without proof a characterization in terms of serial subgroups: *a normal subgroup S of a locally (finite by nilpotent) group G satisfies $S \leq R(G)$ if and only if $\langle H, S \rangle \text{ ser } G$ whenever $H \text{ ser } G$.*

The next lemma, required later, is no doubt well known and is rather similar to Lemma 4(b): indeed it seems quite possible that the two results may have a common generalization.

LEMMA 5. *If G is a polycyclic by finite group and S is a normal subgroup of G such that every chief factor X/Y of G with $X \leq S$ is a central factor of G , then S is contained in the hypercentre of G .*

Proof. If S is finite, the result is clear. If S is infinite, then S will contain a free Abelian normal subgroup $A \neq 1$ of G which is rationally irreducible as a G -module. For each prime p , there will be a maximal G -subgroup B_p satisfying $A^p \leq B_p < A$, and, from our hypothesis, each A/B_p will be a central factor of G . Thus $[G, A] \leq \bigcap B_p$, and $[G, A]$ is a G -subgroup having infinite index in A , so that $[G, A] = 1$. Therefore A lies in the centre of G . An easy induction on the Hirsch length of S now completes the proof of Lemma 5.

Our final result concerning Engel subsets is one that was mentioned in the Introduction.

LEMMA 6. *If G is a \mathfrak{B} -group, then $R(G)$ is a locally nilpotent subgroup of G .*

Proof. Let x_1, \dots, x_n be a finite set of elements of $R(G)$ and let $w \in G$. If the group $D = \langle x_1, \dots, x_n, w \rangle$ is not nilpotent, then, because $G \in \mathfrak{B}$, there is a finite non-nilpotent quotient group D/K . Now $Kx_i \in R(D/K)$ for each i , and

$R(D/K)$ is the hypercentre of D/K , so that D/K is generated by its hypercentre and Kw and therefore must be nilpotent. This contradiction proves that D is nilpotent. We conclude that any finite subset of $R(G)$ generates a nilpotent subgroup, and (taking $n = 2$) that $R(G)$ is a subgroup.

2.2. Here we describe the groups in the class Δ of centreless finite groups each of whose subgroups is a subnormal subgroup or is nilpotent. We begin with

LEMMA 7. *Let G be a finite group each of whose subgroups either is a subnormal subgroup or is nilpotent. For some prime p , G has a normal Sylow p -subgroup and a nilpotent p -complement.*

Proof. We may obviously assume that G is not nilpotent. A theorem of Šmidt [28] asserts that a finite group all of whose proper subgroups are nilpotent is necessarily soluble. Thus G has no non-Abelian simple sections, and so is soluble. Therefore G has a unique conjugacy class \mathcal{C} of self-normalizing nilpotent subgroups (cf. Carter [5]), and because G is non-nilpotent, each member of \mathcal{C} is a proper subgroup. Let P be a Sylow p -subgroup contained in no member of \mathcal{C} . Because $N_G(P)$ is self-normalizing, it can be neither a proper subnormal subgroup of G nor a nilpotent subgroup. It follows that $N_G(P) = G$ and that $P \triangleleft G$.

Let $C \in \mathcal{C}$. If $x \in N_G(PC)$, then C and C^x are self-normalizing nilpotent subgroups of PC and so are conjugate in PC ; and it follows that $x \in PC$ and that $N_G(PC) = PC$. Since PC is certainly not nilpotent, we conclude that $PC = G$. If Q is a p -complement in C , then Q is nilpotent and also is a p -complement in G , and Lemma 7 follows.

Now we can prove

PROPOSITION 2. (a) *A finite group $G \neq 1$ is a Δ -group if and only if it is generated by an Abelian minimal normal subgroup P and a cyclic subgroup $Q \neq 1$ such that (i) $C_Q(P) = 1$ and (ii) every non-trivial subgroup of Q acts irreducibly on P by conjugation.*

(b) *If $1 \neq G \in \Delta$ and $H \leq G$, then either H is cyclic or H is a subnormal subgroup (of defect at most two) of G ; if H is not Abelian, then $H \in \Delta$, and the derived groups of G and H coincide.*

Proof. Suppose $1 \neq G \in \Delta$. By Lemma 7, G has a normal Sylow p -subgroup P and a nilpotent p -complement Q for some prime p . Let K be a minimal normal subgroup of G . We cannot have $K \cap P = 1$, for then K would be G -isomorphic to PK/P and so would be central in G . Thus K is a p -group and $K \leq P$. If KQ were nilpotent, we would have $[K, Q] = 1$, and K would be a minimal normal subgroup of P , so that K would again be central in G . Therefore KQ is subnormal in G , and, because KQ/K is a Hall subgroup of G/K , we have in fact $KQ \triangleleft G$.

It follows that $[P, Q] \leq K$, and, because $G/[P, Q]$ is nilpotent, that $[P, Q] = K$. Thus G is monolithic, with monolith $[P, Q]$. Moreover P must be an elementary Abelian group, or Q would act trivially on its Frattini factor group, and so, by a theorem of Burnside (cf. Gorenstein [15, Theorem 5.1.4]), would centralize P , a contradiction. Because $P = [P, Q]C_p(Q)$ (cf. Gorenstein [15, Theorem 5.2.3]) and because $C_p(Q)$ is central in G , we have $P = [P, Q]$, so that Q acts irreducibly on P .

We assert that every non-trivial subgroup of Q acts irreducibly on P . If this is not the case, we may find a subgroup Q_1 of Q which (a) acts irreducibly on P and (b) has a maximal subgroup $Q_2 \neq 1$ which does not act irreducibly on P . Let $P_1 \neq 1$ be a proper Q_2 -invariant subgroup of P . By Maschke's Theorem, we may write $P = P_1 \times P_2$ where P_2 is also Q_2 -invariant. If for $i = 1$ or 2 we have $P_i Q_2$ subnormal in G , then $P_i Q_2 / P_i$ is a subnormal Hall subgroup of PQ_2 / P_i , so that $[P, Q_2] \leq P_i$. Because $[P, Q_2]$ is Q_1 -invariant, it follows that $[P, Q_2] = 1$. Thus $C_Q(P)$ is a non-trivial normal subgroup of Q and has non-trivial intersection U with the centre of Q ; and because U is then central in G we have a contradiction. Therefore $P_1 Q_2$ and $P_2 Q_2$ are both nilpotent, and so $[P_1, Q_2] = [P_2, Q_2] = 1$. However this again implies that $C_Q(P)$ is non-trivial, and a further contradiction ensues. Our assertion follows.

Now let A be the centre of Q . Then P may be regarded as an irreducible $Z_p A$ -module, and Q may be embedded in its centralizer algebra D . By Schur's Lemma, D is a division ring, and because D is finite, Wedderburn's Theorem implies that D is in fact a field. Therefore Q may be embedded in the multiplicative group of a field and so is cyclic. We have shown that the Δ -group G has the structure claimed for it in statement (a) of Proposition 2.

Now we assume instead that G is generated by an Abelian minimal normal p -subgroup P and a cyclic subgroup Q such that $C_Q(P) = 1$ and such that every non-trivial subgroup of Q acts irreducibly on P , and we let H be a subgroup of G . We show that one of the following holds: $P \leq H$, or $H \leq P$, or H is conjugate to a subgroup of Q . Statement (b) and the remaining implication of (a) follow immediately from this.

Suppose then that $P \not\leq H$ and $H \not\leq P$. Since $H \cap P$ is a Sylow p -subgroup of H , we have $H = (H \cap P)S$, where S is non-trivial and is a p -complement in H . The subgroup S is contained in a p -complement in G , and some conjugate S^x is therefore contained in Q . Since $(H \cap P)^x$ is invariant under the non-trivial subgroup S^x of Q , we have either $H \cap P = P$ or $H \cap P = 1$. Thus either $H = PS > P$, a contradiction, or $H = S \leq Q^{x^{-1}}$, as required. This completes the proof of Proposition 2.

2.3. We conclude Section 2 by recording a number of well-known results in forms tailored to our needs.

LEMMA 8. *If G is a finite extension of a locally nilpotent normal subgroup K ,*

and if K is not a Černikov group, then there is an infinite descending chain (K_i) of normal subgroups of G contained in K .

Proof. Lemma 8 follows immediately from the fact that the minimal condition for normal subgroups is inherited by normal subgroups of finite index (Wilson [35, Theorem A]) and the fact that locally nilpotent groups satisfying the minimal condition for normal subgroups are Černikov groups (McLain [22, Theorem 3.1]).

LEMMA 9. *Let G be a locally finite group.*

- (a) *If all subgroups of G are serial subgroups, then G is locally nilpotent.*
- (b) *Suppose that H is a serial subgroup of G . If H is locally nilpotent, then so is its normal closure H^G in G ; if instead H is a π -group for a set of primes π , then H^G is also a π -group.*
- (c) *Suppose that G is generated by a locally nilpotent normal q' -subgroup K and a cyclic q -subgroup $\langle x \rangle$, for some prime q . If L is a normal subgroup of G contained in K , and if $L\langle x \rangle$ is a serial subgroup of G , then $[K, \langle x \rangle] \leq L$.*

Proof. Assertions (a) and (b) are immediate consequences of corresponding assertions about finite groups and their subnormal subgroups. In (c), we have $L\langle x \rangle/L \text{ ser } G/L$; and $L\langle x \rangle/L$ is also a Sylow q -subgroup of G/L . We conclude from (b) that $L\langle x \rangle/L \triangleleft G/L$, and assertion (c) follows.

LEMMA 10. *Let G be a locally finite group.*

- (a) *The Hirsch–Plotkin radical of G centralizes each chief factor of G .*
- (b) *Suppose that G is generated by a locally nilpotent normal subgroup K and a cyclic q -group, for some prime q . If L is a normal subgroup of G contained in K , then $R(L\langle x \rangle) \cap K \leq R(G)$. If $R(G) = 1$, then K is a q' -group.*

Proof. Assertion (a) is well known (cf. Theorem 1.B.10 of Kegel and Wehrfritz [20]). In (b), we let X/Y be a chief factor of G with $X \leq R(L\langle x \rangle) \cap K$. From (a), we have $C_G(X/Y) \geq K$, so that X/Y is also a chief factor of $L\langle x \rangle$, and indeed, by Lemma 4, a central factor of $L\langle x \rangle$. It follows that every chief factor of G below $R(L\langle x \rangle) \cap K$ is a central factor of G , and, again from Lemma 4, we conclude that $R(L\langle x \rangle) \cap K \leq R(G)$. To prove the last part of (b), we take for L the Sylow q -subgroup of K . Then $L\langle x \rangle$ is a locally finite q -group, and so $L \leq R(L\langle x \rangle) \cap K$. It now follows from what we have just proved that $L \leq R(G) = 1$, and the proof of Lemma 10 is complete.

3. THEOREM A

3.1. We begin with a local theorem which provides for important reductions in the proof of Theorem A. We prove

PROPOSITION 3. *Suppose that G is a group having a local system \mathcal{L} with the following properties:*

- (a) *$R(H)$ is a locally nilpotent (normal) subgroup of H for each $H \in \mathcal{L}$, and*
- (b) *$H/R(H) \in \Delta$ for $H \in \mathcal{L}$.*

Then $R(G)$ is a locally nilpotent (normal) subgroup of G , and $G/R(G) \in \Delta$.

Proof. If $H = R(H)$ for all $H \in \mathcal{L}$, then it is rather easy to see that $G = R(G)$. We therefore assume that there is a subgroup $H_0 \in \mathcal{L}$ with $R(H_0) < H_0$. Suppose $H_1, H_2 \in \mathcal{L}$ satisfy $H_0 \leq H_1 \leq H_2$. First we note that

$$H_0 \cap R(H_1) \leq R(H_0) < H_0,$$

so that $R(H_1) < H_1$. Next, we have

$$H_1 \cap R(H_2) \leq R(H_1),$$

so that $H_1/R(H_1)$ is a homomorphic image of $H_1/(H_1 \cap R(H_2))$, which is isomorphic to $H_1R(H_2)/R(H_2)$. Thus, because $H_1/R(H_1)$ is non-Abelian, so is $H_1R(H_2)/R(H_2)$; and it follows from Proposition 2 that $H_1R(H_2)/R(H_2)$ is a Δ -group and that each proper homomorphic image of $H_1R(H_2)/R(H_2)$ is cyclic. We conclude that

$$R(H_1) = H_1 \cap R(H_2).$$

From this it follows very easily that the subgroups $R(H)$ with $H_0 \leq H \in \mathcal{L}$ form a local system for a normal subgroup R , and that $R = R(G)$. Moreover it follows that

$$R(H) = H \cap R$$

whenever $H_0 \leq H \in \mathcal{L}$. For any such H , we therefore have

$$H_0/R(H_0) \cong H_0R(H)/R(H) \leq H/R(H),$$

so that, by Proposition 2, the derived groups of $H_0/R(H_0)$ and $H/R(H)$ have the same order, p^n say, where p is a prime, and the order of $H/R(H)$ is bounded (certainly by $p^n | \text{GL}_n(p) |$) in terms of the structure of $H_0/R(H_0)$. Because $HR/R \cong H/R(H)$, it follows that the finitely generated subgroups of G/R have bounded orders, and that G/R is finite. Thus $G = HR$ for some $H \in \mathcal{L}$, and $G/R \cong H/R(H) \in \Delta$, as required.

It is worth noting that Propositions 1, 2 and 3 yield immediately a characterization of locally finite groups each of whose subgroups either is a serial subgroup or is locally nilpotent:

PROPOSITION 4. *Let G be a locally finite group. Each subgroup of G either is a serial subgroup or is locally nilpotent if and only if $G/R(G) \in \mathcal{A}$.*

The requirement here that G be locally finite can be relaxed to the requirement that G be a \mathfrak{W} -group by the use of the techniques of Section 3.2 below, and of course the whole of this extended form of Proposition 4 may be read off from the very much harder Theorem A, except in the case in which G is a Černikov group.

3.2. With the aid of a very simple preliminary lemma, we can now show that \mathfrak{W} -groups which are not locally finite behave in accordance with Theorem A.

LEMMA 11. *If G is a finitely generated \mathfrak{W} -group satisfying the minimal condition for non-serial non-locally-nilpotent subgroups, then G is nilpotent by finite.*

Proof. Suppose the Lemma is false; then, since G is a non-nilpotent \mathfrak{W} -group, G has non-nilpotent finite images and therefore has subgroups of finite index which are not serial (and not locally nilpotent). Let G_0 be a minimal such subgroup. Because G_0 is again a finitely generated non-nilpotent \mathfrak{W} -group, it will have subgroups of finite index which are not serial in G_0 , and therefore not serial in G . The Lemma follows.

We have to prove the implication (a) \Rightarrow (b) of Theorem A: that a \mathfrak{W} -group G satisfying the minimal condition for non-serial non-locally-nilpotent subgroups either is a Černikov group or is a locally (finite by nilpotent) group such that $R(G)$ is a locally nilpotent normal subgroup with $G/R(G) \in \mathcal{A}$. Lemma 11 shows that G is locally (nilpotent by finite); thus, unless G is locally finite, it will have a local system of finitely generated infinite nilpotent by finite subgroups, and, by Proposition 3, the above conclusion will follow if we establish

THEOREM A1. *If G is an infinite finitely generated nilpotent by finite group satisfying the minimal condition for non-serial, non-locally-nilpotent subgroups, then $R(G)$ is nilpotent and $G/R(G) \in \mathcal{A}$; moreover G is finite by nilpotent.*

We now prove this result, deferring for the moment the rather more difficult case of locally finite groups. We note that it is only necessary to prove that $G/R(G) \in \mathcal{A}$, for then $R(G)$ will be the hypercentre of G (cf. Baer [4, p. 257]); thus $R(G)$ will be nilpotent, and also, by a theorem of Baer [1], G will be finite by nilpotent. With the hypothesis of Theorem A1, we prove first

(a) *each subgroup of finite index in G either is nilpotent or is a subnormal subgroup.*

If this is not the case, then among the subgroups of finite index which neither are nilpotent nor are subnormal in G , we can find a minimal one H . Because H is not nilpotent, it has a finite non-nilpotent image, and there will be a normal

subgroup N of finite index in G , contained in H , such that H/N is not nilpotent. Because G is nilpotent by finite, we may choose N to be nilpotent. Let p be a prime not dividing $|G/N|$, and let M be the subgroup generated by the derived group of N and all p th powers of elements of N . Then $M \triangleleft G$ and $M < N$. By the Schur-Zassenhaus Theorem, N/M will have a complement X/M in H/M . Clearly X/M is isomorphic to H/N , so that X is not nilpotent. Moreover, if X were subnormal in G , so would be its join H with the normal subgroup N , and this is not the case. We therefore have a contradiction to the minimal choice of H , and (a) follows.

We may obviously assume that G is not nilpotent. Thus, G has a non-nilpotent finite image, and therefore, by (a) and the definition of the class \mathcal{A} , G has a non-trivial quotient group $G/G_0 \in \mathcal{A}$. Next we prove

(b) *every chief factor of G below G_0 is a central factor of G .*

Let A/B be a chief factor of G below G_0 , and write bars for factor groups modulo B . Since \bar{G} is a finitely generated nilpotent by finite group, \bar{A} is finite and G is residually finite; thus there is a normal subgroup \bar{T} of finite index in \bar{G} such that $\bar{T} \leq \bar{G}_0$ and such that $\bar{T} \cap \bar{A} = 1$. From (a) and the fact that proper homomorphic images of \mathcal{A} -groups are cyclic (cf. Proposition 2) we conclude that \bar{G}_0/\bar{T} is the hypercentre of \bar{G}/\bar{T} . Thus the minimal normal subgroup $\bar{A}\bar{T}/\bar{T}$ of \bar{G}/\bar{T} is a central factor of \bar{G}/\bar{T} , and $[\bar{A}, \bar{G}] \leq \bar{A} \cap \bar{T} = 1$, as required.

We may now apply Lemma 5, to deduce from (b) that G_0 lies in the hypercentre of G . Because G/G_0 has trivial centre and because the hypercentre and set of right Engel elements of a Noetherian group coincide, it follows that $G_0 = R(G)$, and Theorem A1 is proved.

3.3. Next we begin the study of locally finite groups satisfying the minimal condition for non-serial non-locally-nilpotent subgroups. If there is such a group G which neither is a Černikov group nor satisfies $G/R(G) \in \mathcal{A}$, then each subgroup of G either is a serial subgroup or is locally nilpotent, or there are non-serial non-locally-nilpotent subgroups. In the latter case, we may replace G by a non-serial non-locally-nilpotent subgroup H minimal with respect to neither being a Černikov group nor satisfying $H/R(H) \in \mathcal{A}$. In each case, we are concerned with a member of the class Γ of locally finite groups G which satisfy the minimal condition for non-serial non-locally-nilpotent subgroups and each of whose subgroups X satisfies one of (a) X ser G , (b) X is a Černikov group or (c) $X/R(X) \in \mathcal{A}$. Thus the proof of Theorem A will be complete if we establish

THEOREM A2. *If $G \in \Gamma$, then either G is a Černikov group or $G/R(G) \in \mathcal{A}$.*

In the balance of this section we shall prove this result under the additional hypothesis that G has no infinite simple sections.

LEMMA 12. *If $G \in \Gamma$ and if G is an extension of a locally nilpotent group by a cyclic group of prime power order, then $G/R(G)$ is a Černikov group.*

Proof. We may assume $R(G) = 1$ and write $G = H\langle x \rangle$, where H is a locally nilpotent normal subgroup of G and where $H\langle x \rangle/H$ has order a power of a prime q . Replacing x by one of its powers if necessary, we may suppose that x has q -power order. It follows from Lemma 10(b) that H is a q' -group; moreover, if we set $L = [H, \langle x \rangle]$, it follows from an easy extension of Theorem 5.3.6 of Gorenstein [15] that $L = [L, \langle x \rangle]$. First we prove

(a) *L is a Černikov group.*

If this is not the case, then, because $L\langle x \rangle$ is locally nilpotent by finite, there is by Lemma 8 an infinite descending chain (K_i) of normal subgroups of $L\langle x \rangle$ contained in L . Because $\langle x \rangle$ is finite, the chain $(K_i\langle x \rangle)$ is also infinite. It follows that, for some K_i satisfying $1 < K_i < L$, either $K_i\langle x \rangle \text{ ser } G$ or $K_i\langle x \rangle$ is locally nilpotent. In the former case, we have $K_i\langle x \rangle \text{ ser } L\langle x \rangle$ and $[L, \langle x \rangle] \leq K_i$ by Lemma 9(c); in the latter, we have $[K_i, \langle x \rangle] = 1$ and

$$K_i \leq R(L\langle x \rangle) \cap L \leq R(G)$$

by Lemma 10(b). In each case we have a contradiction, and (a) follows.

Now G/L is locally nilpotent, and $L\langle x \rangle/L$ is a Sylow q -subgroup of G/L , so that $L\langle x \rangle \triangleleft G$. Because $\langle x \rangle$ is a finite Sylow subgroup of $L\langle x \rangle$, an easy and well known extension of the Frattini argument yields

$$(b) \quad G = (L\langle x \rangle) N_G(\langle x \rangle) = L N_G(\langle x \rangle).$$

If L is finite, then (b) implies that $C_G(x)$ has finite index in G ; thus the intersection D of its normal interior in G with H is a normal subgroup of finite index in G and satisfies $[D, \langle x \rangle] = 1$. From Lemma 10(b) we conclude that $D \leq R(G) = 1$ and that G is finite. We suppose then that L is infinite. Because of (a) and (b), our proof will be complete if we can establish

(c) *$N_G(\langle x \rangle)$ is a Černikov group.*

If (c) is false, then there will be an infinite descending chain

$$N_G(\langle x \rangle) > U_1 > \cdots > U_n > \cdots > \langle x \rangle.$$

Let S be a finite non-trivial (necessarily proper) G -subgroup of L . For some i , either $SU_i \text{ ser } G$ or SU_i is locally nilpotent. If $SU_i \text{ ser } G$, then because $S\langle x \rangle \triangleleft SU_i$ we have $S\langle x \rangle \text{ ser } G$, so that G/S is locally nilpotent. If on the other hand SU_i is locally nilpotent, then so is $S\langle x \rangle$, and $S \leq R(G) = 1$. These contradictions show that $N_G(x)$ must be a Černikov group, and Lemma 12 is proved.

LEMMA 13. *If $G \in \Gamma$ and if G is a finite extension of a locally nilpotent normal subgroup H , then either G is a Černikov group or $G/R(G) \in \Delta$.*

Proof. Let \mathcal{T} be the set of subgroups T of G such that $T \geq H$ and such that T/H is cyclic of prime power order. For each $T \in \mathcal{T}$, the group $T/R(T)$ is a Černikov group by Lemma 12; and because \mathcal{T} is finite, G/N is a Černikov group, where $N = \bigcap (R(T); T \in \mathcal{T})$.

We assert that $N \leq R(G)$. Let L/M be a chief factor of G below N . By Lemma 10(a), $H \leq C_G(L/M)$. Thus L/M is locally nilpotent and is a minimal normal subgroup of its split extension by G/H , so that L/M is a finite Abelian p -group, for some prime p . Let T be an element of \mathcal{T} such that $|T/H|$ is not a power of p . Because $L \leq R(T)$, T/H acts nilpotently on L/M , and it follows from Theorem 5.3.2 of Gorenstein [75] that T/H centralizes L/M . Therefore $G/C_G(L/M)$ is a finite p -group, and L/M is a minimal normal p -subgroup of its extension by $G/C_G(L/M)$, so that $G/C_G(L/M)$ must actually be trivial, and L/M must be a central factor of G . Because this argument holds for each chief factor L/M with $L \leq N$, it follows from Proposition 1 that $N \leq R(G)$, as asserted above.

So far we have proved that $G/R(G)$ is a Černikov group. We now assume that G is not a Černikov group, and will prove that $G/R(G) \in \Delta$. Suppose that X is a finite non-nilpotent subgroup of G . Since G is not a Černikov group while $G/R(G)$ is a Černikov group, the subgroup $R(G)X$ cannot be a Černikov group, so that, by Lemma 8, there is an infinite descending chain (R_i) of X -subgroups of $R(G)$. Because X is finite and non-nilpotent, the chain $(R_i X)$ is an infinite descending chain of non-locally-nilpotent subgroups. Thus, $R_i X \text{ ser } G$ for some i , and, by Proposition 1, $R(G)X \text{ ser } G$. It follows that each finite subgroup of $G/R(G)$ either is a serial subgroup or is nilpotent, and therefore, by Lemma 1 and Proposition 3 (with \mathcal{L} taken as the set of finite subgroups of $G/R(G)$), that $G/R(G) \in \Delta$. This completes the proof of Lemma 13.

LEMMA 14. *If $G \in \Gamma$ and if G is an extension of its Hirsch–Plotkin radical H by a locally nilpotent group, then either G is a Černikov group or $G/R(G) \in \Delta$.*

Proof. If H is a Černikov group, then so also is $G/C_G(H)$ by a result of Baer [3], (cf. Kegel and Wehrfritz [20, p. 35]); and, because $C_G(H)$ is a locally nilpotent normal subgroup of G , we conclude that $C_G(H) \leq H$ and that G is a Černikov group. If H is not a Černikov group, we write \mathcal{L} for the local system of subgroups $K \geq H$ with K/H finite, and use Lemma 13 and Proposition 3 to deduce that $G/R(G) \in \Delta$.

LEMMA 15. *Let G be a Γ -group with finite Hirsch–Plotkin radical H . If G has no infinite simple sections, then G is finite.*

Proof. We suppose instead that G is infinite. Let K be any infinite normal

subgroup of G and let $K < L \leq G$. Clearly L can neither be a Černikov group nor satisfy $L/R(L) \in \mathcal{A}$, so that L is not serial. Thus every subgroup of G/K is a serial subgroup, and appealing to Lemma 9(a) we deduce that G/K is locally nilpotent. Writing U for the intersection of all the infinite normal subgroups of G , we therefore have G/U locally nilpotent. If U were finite, then its centralizer in G would be an infinite locally nilpotent normal subgroup, and we would have a contradiction. Thus U is an infinite group each of whose proper G -subgroups N is finite. For each such N the group $G/C_G(N)$ is finite, so that $U \leq C_G(N)$. Therefore U cannot be the join of its proper G -subgroups, because then it would be both infinite and Abelian, in contradiction to our hypothesis. It follows that the join J of all proper G -subgroups of U is finite and central in U , and that $X = U/J$ is an infinite chief factor of G .

Our hypothesis implies that X cannot be simple. Thus, there is a subgroup $Y \triangleleft X$ such that $1 < Y < X$. Replacing Y by $C_X(Y)$ if necessary, we may suppose Y infinite. We write $M/J = Y$. The Hirsch–Plotkin radical of M cannot be infinite, for then its normal closure in G would be an infinite locally nilpotent normal subgroup of G . Thus every subgroup of X containing Y must be a serial subgroup of X and, by Lemma 9(a), X/Y must be locally nilpotent. Since X/Y is isomorphic to U/M and to each group U/M^g with $g \in G$, and since $\bigcap (M^g; g \in G) = J$, it follows that U/J is locally nilpotent. Finally, because J lies in the centre of U , we conclude that the infinite normal subgroup U is locally nilpotent, and this contradiction finishes the proof of Lemma 15.

The main result of this section now follows easily:

LEMMA 16. *If G is a Γ -group having no infinite simple sections, then either G is a Černikov group or $G/R(G) \in \mathcal{A}$.*

Proof. Let H be the Hirsch–Plotkin radical of G . By Lemma 14, the Hirsch–Plotkin radical of G/H is finite; thus, by Lemma 15, G/H is itself finite. The result now follows from Lemma 13.

3.4. Because the class Γ is section-closed, Lemma 16 reduces the proof of Theorem A2 (and therefore of Theorem A) to the proof of the following lemma:

LEMMA 17. *There are no infinite simple Γ -groups.*

We begin with two preparatory results, the first being rather elementary.

LEMMA 18. *If G is an infinite simple Γ -group then each subgroup of G is a serial subgroup or a Černikov group or a locally nilpotent group.*

Proof. Certainly every subgroup H which is neither a serial subgroup nor a Černikov group satisfies $H/R(H) \in \mathcal{A}$; and for such an H we may write

$H = R(H)X$ where X is a finite subgroup. If X is nilpotent, then H is locally nilpotent by Proposition 1, as required. Otherwise, by Lemma 8, there is an infinite descending chain (R_i) of X -subgroups of $R(H)$, and the chain $(R_i X)$ is an infinite chain of non-locally-nilpotent subgroups. Thus $R_i X \text{ ser } G$ for some i , and $R_i \text{ ser } G$. Since R_i is a locally nilpotent serial subgroup, its normal closure R_i^G is locally nilpotent, by Lemma 9(b); and since R_i is non-trivial and G is assumed to be infinite and simple, a contradiction follows.

The second preliminary lemma prepares the way by showing that certain specific infinite simple locally finite groups are not Γ -groups. These are the projective special linear groups $\text{PSL}_2(F)$ over infinite locally finite fields F and the Suzuki groups $\text{Sz}(F)$ over certain infinite locally finite fields F of characteristic two. The groups $\text{Sz}(F)$ for infinite F are defined in much the same way as for finite F ; they are discussed in Kegel [18] and in Chapter 4 of [20].

LEMMA 19. *No group of type $\text{PSL}_2(F)$ or of type $\text{Sz}(F)$, where F is an infinite locally finite field, is a Γ -group.*

Proof. To show that a simple non-Abelian group G is not a Γ -group, it suffices because of Lemma 18 to exhibit in G a metabelian subgroup H which is neither locally nilpotent nor a Černikov group; H could not be a serial subgroup, for its derived group would then be a non-trivial locally nilpotent serial subgroup, and Lemma 9(b) would yield a contradiction.

In the case of $\text{PSL}_2(F)$, we may take for H the image in $\text{PSL}_2(F)$ of the group of lower triangular matrices in $\text{SL}_2(F)$. It is rather easy to check that H is metabelian and is neither locally nilpotent nor a Černikov group.

The group $\text{Sz}(F)$ (for a suitable field F) is a subgroup of $\text{SL}_4(F)$ defined in terms of an automorphism θ of F satisfying

$$f^{\theta^2} = f^2 \quad (f \in F),$$

and is generated by (a) a group of lower unitriangular matrices, (b) the group D of diagonal matrices

$$\text{diag}(f^{1+\theta^{-1}}, f^{\theta^{-1}}, f^{-\theta^{-1}}, f^{-1-\theta^{-1}})$$

with $0 \neq f \in F$, and (c) the permutation matrix

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(cf. Suzuki [32, p. 133]). It is trivial to verify that τ is an involution inverting every element of D . Thus the group $\langle D, \tau \rangle$ is metabelian and is neither locally nilpotent nor a Černikov group. Lemma 20 follows.

3.5. We come now to the proof of Lemma 17. In the course of this proof, frequent reference will be made to results in the book by Kegel and Wehrfritz [20]. We suppose that Lemma 17 is false and that G is an infinite simple Γ -group. First we note that G has involutions: otherwise, by the Feit–Thompson theorem [14], G would be locally soluble and could not be simple (cf. [20, p. 11]). Moreover, the extension of the Brauer–Fowler theorem to infinite locally finite groups (Šunkov [29], Theorem 3; cf. [20, p. 105]) guarantees that the centralizer in G of each involution is infinite. We shall use repeatedly and without further explanation the fact that all non-serial subgroups of G are Černikov groups or locally nilpotent groups (from Lemma 18) and that G has no non-trivial locally nilpotent serial subgroups (from Lemma 9(b)).

We begin by proving

- (a) $C_G(i)$ is a non-locally-nilpotent Černikov group, for some involution i of G .

Certainly we cannot have $C_G(i) \text{ ser } G$ for any i , because this would imply $\langle i \rangle \text{ ser } G$. Thus, if (a) is false, then the centralizer of every involution is locally nilpotent, and Theorem 4.27 of Kegel and Wehrfritz [20] implies that G is isomorphic to either $\text{PSL}_2(F)$ or $\text{Sz}(F)$ for a suitable locally finite field F . Lemma 19 yields a contradiction, and (a) follows. Moreover, (a), combined with a result of Šunkov [30] (cf. [20, p.82]) implies

- (b) all 2-subgroups of G are Černikov groups.

We now apply Theorem 1 of Wehrfritz [34]; this theorem implies that if S is a simple locally finite group such that $C_S(i)$ has a unique maximal divisible 2-subgroup of finite rank for each involution $i \in S$, then either S is isomorphic to a group $\text{PSL}_2(F)$ or the divisible 2-subgroups of S are in fact trivial. Because the centralizer of each involution of our group G either is locally nilpotent or is a Černikov group, the hypothesis of Wehrfritz' Theorem is satisfied, and an appeal to Lemma 19 yields

- (c) all 2-subgroups of G are finite.

Our hypothesis on G and assertion (c) yield immediately

- (d) $C_G(i)$ is a finite extension of a 2'-group, for each involution i of G .

Next we establish

- (e) every non-serial subgroup of G containing involutions (in particular, every centralizer of an involution) is a Černikov group.

If (e) is false, then there will be non-Černikov locally nilpotent subgroups containing involutions, and, by Zorn's Lemma, there will be a maximal such

subgroup H . The Sylow 2-subgroup D of H is non-trivial and $N_G(D) \geq H$, so that the maximality of H implies $N_G(D) = H$. Thus, if P were a (necessarily finite) 2-subgroup of G with $D < P$, we would have $D < N_P(D) \leq H$, a contradiction. It follows that D is a Sylow 2-subgroup of G . Of course the centralizer of each involution of D contains the Hall 2'-subgroup of H , so cannot be a Černikov group, and must therefore be locally nilpotent. Because G has a finite Sylow 2-subgroup, all of its Sylow 2-subgroups are finite and conjugate (cf. [20, p. 25]), and we conclude that the centralizer in G of any involution is locally nilpotent, in contradiction to (a). Assertion (e) follows.

We write X^0 for the least subgroup of finite index in a Černikov group X , and prove

(f) *for each involution i of G there is a (non-trivial) divisible Abelian subgroup $X(i)$ maximal subject to containing $(C_G(i))^0$ and being normalized by i . $X(i)$ contains all other divisible Abelian groups which contain $(C_G(i))^0$ and are normalized by i , and $(N_G(X(i)))^0 = X(i)$.*

Because $C_G(i)$ is an infinite Černikov group, there certainly are non-trivial divisible Abelian subgroups which contain $(C_G(i))^0$ and are normalized by i ; and since a union of a chain of subgroups with these properties again has these properties, the existence of a maximal such subgroup $X(i)$ follows. If Y is now a divisible Abelian group containing $(C_G(i))^0$ and satisfying $Y^i = Y$, then

$$1 < (C_G(i))^0 \triangleleft \langle X(i), Y, i \rangle,$$

so that, from assertion (e), $\langle X(i), Y, i \rangle$ is a Černikov group. Thus $\langle X(i), Y \rangle$ is a divisible Abelian subgroup normalized by i , and the maximality of $X(i)$ implies that $Y \leq X(i)$. Finally, because $i \in N_G(X(i))$, it follows from (e) that $N_G(X(i))$ is a Černikov group, and from the maximality of $X(i)$ that $(N_G(X(i)))^0 = X(i)$. Assertion (f) is therefore proved.

We now want to show that G satisfies the hypotheses of Theorem 5.5 of Kegel and Wehrfritz [20]; the conclusion will then be that G is of type $\text{PSL}_2(F)$ for some locally finite field F , and Lemma 19 will provide a final contradiction. We have already verified some of the hypotheses of this theorem in assertions (c) and (d) above. The remaining hypothesis is that G should have a family \mathcal{E} of infinite Abelian subgroups such that

(α) for all $X \in \mathcal{E}$ and all $g \in G$, one has $X^g \in \mathcal{E}$,

(β) if for $X \in \mathcal{E}$ and an involution $i \in N_G(X)$ one has $C_G(i) \not\leq N_G(X)$, then $C_X(i)$ is finite, and

(γ) for every involution i of G there is a unique subgroup $X \in \mathcal{E}$ such that $C_G(i) \leq N_G(X)$.

For \mathcal{E} we take the family of subgroups $X(i)$, with i running through all the involutions in G . Obviously \mathcal{E} is not empty and condition (α) above is satisfied. If i is an involution and $g \in C_G(i)$, then $(X(i))^g = X(i^g) = X(i)$, so that $g \in N_G(X(i))$. Thus $C_G(i) \leq N_G(X(i))$. If X is a member of \mathcal{E} such that $C_G(i) \leq N_G(X)$, then $(C_G(i))^\circ \leq X$ by assertion (f), and i normalizes X , so that (again by assertion (f)), we have $X \leq X(i)$. Therefore, because $X(i)$ is Abelian, we also have $X(i) \leq (N(X))^\circ = X$, and hence $X = X(i)$. Thus condition (γ) above is satisfied. Now if X is a member of \mathcal{E} such that $C_X(i)$ is infinite and $i \in N_G(X)$, then

$$1 < (C_X(i))^\circ \triangleleft \langle (C_G(i))^\circ, X, i \rangle,$$

so that, by (e), $\langle (C_G(i))^\circ, X, i \rangle$ is a Černikov group. It follows from (f) that

$$\langle (C_G(i))^\circ, X \rangle \leq \langle (C_G(i))^\circ, X, i \rangle^\circ \leq X(i),$$

and that $X \leq X(i)$. Thus $X(i) \leq (N_G(X))^\circ = X$, and $X = X(i)$, so that condition (β) above holds. We have shown that all hypotheses of Theorem 5.5 of [20] are satisfied, and the resulting contradiction completes our proof of Lemma 17.

4. THEOREM B AND THEOREM C

4.1. To complete the proof of Theorem C(i) we have to show that if G is a \mathfrak{B} -group satisfying the minimal condition for non-serial non-Abelian subgroups, and if G is neither a Černikov group nor locally nilpotent, then the central factor group of G is in \mathcal{A} . From Theorem A we know that G is locally (finite by nilpotent) and that $1 \neq G/R(G) \in \mathcal{A}$; it therefore suffices to show that $[G, R(G)] = 1$. We treat the cases of groups which are locally finite and groups which are not locally finite separately.

Case 1. G is locally finite. Suppose $[G, t] \neq 1$ for some $t \in R(G)$. Let g_1 be any element of G with $[g_1, t] \neq 1$, and write $X = \langle g_1, t \rangle$. Because G is not a Černikov group and $R(G)$ is locally nilpotent, it follows from Lemma 8 that there is an infinite descending chain (R_i) of G -subgroups of $R(G)$; and because X is finite and non-Abelian, the chain $(R_i X)$ is an infinite chain of non-Abelian subgroups of G . Thus, $R_i X$ ser G for some i , and by Proposition 1 we have

$$R(G)\langle g_1 \rangle = R(G)X = R(G)(R_i X) \text{ ser } G.$$

If $g_2 \in G$ and $[g_2, t] = 1$, then $[g_1 g_2, t] \neq 1$, and the argument above (with $g_1 g_2$ in place of g_1) shows that $R(G)\langle g_1 g_2 \rangle$ ser G . It follows that $G/R(G)$ is generated by cyclic subnormal subgroups and so is nilpotent (see Baer [2,

p. 419)], in contradiction to the assumption that $G/R(G)$ is a non-trivial Δ -group.

Case 2. G is not locally finite. In this case, G has a local system \mathcal{L} of infinite finitely generated non-nilpotent subgroups, and if $[G, R(G)] \neq 1$, then there will be an element $t \in R(G)$ with $[G, t] \neq 1$ and a member G_0 of \mathcal{L} containing t . Of course, $t \in R(G_0)$ because $R(G) \cap G_0 \leq R(G_0)$. We choose $g_1 \in G_0$ with $[g_1, t] \neq 1$ and write $X = \langle g_1, t \rangle$. Because G_0 is residually finite and X is non-Abelian, there will be a normal subgroup H of finite index in G_0 , contained in $R(G_0)$, such that XH/H is non-Abelian; moreover, because G_0 is infinite, finitely generated and finite by nilpotent (by Theorem A), G_0 will have an infinite cyclic homomorphic image, and therefore will have an infinite descending chain (K_i) of normal subgroups of finite index co-prime to $|XH/H|$. For each i , we have

$$XH = (XH \cap K_i)H \quad (*)$$

so that each subgroup $XH \cap K_i$ has a homomorphic image isomorphic to XH/H , and so is non-Abelian. Therefore the chain $(XH \cap K_i)$ is an infinite descending chain of non-Abelian subgroups, and its terms must eventually be serial subgroups. Thus, for some i , we have $(XH \cap K_i) \text{ ser } G_0$, so that, from $(*)$ and Proposition 1,

$$R(G_0)\langle g_1 \rangle = R(G_0)X = R(G_0)(XH \cap K_i) \text{ ser } G_0.$$

We now proceed as in the treatment of Case 1; we conclude that $G_0/R(G_0)$ is generated by cyclic subnormal subgroups, and therefore that $G_0/R(G_0) = 1$. The resulting contradiction to the assumption that G_0 is not nilpotent completes the proof of Theorem C(i).

4.2. To prove Theorem C(ii), we must show that if G is a \mathfrak{B} -group satisfying the minimal condition for non-serial subgroups and is not a Černikov group, then G is locally nilpotent. By Theorem A, $R(G)$ is locally nilpotent and $G/R(G) \in \Delta$. We shall prove that each cyclic subgroup of $G/R(G)$ is subnormal in $G/R(G)$, so that $G/R(G)$ is nilpotent; it will follow from this that $G = R(G)$ and that G is locally nilpotent.

Let $A/R(G)$ be a cyclic subgroup of $G/R(G)$, and write $A = R(G)\langle x \rangle$. If $\langle x \rangle$ has infinite order, we choose a prime p not dividing $|G : R(G)|$ and consider the strictly descending chain

$$\langle x \rangle > \langle x^p \rangle > \cdots > \langle x^{p^n} \rangle > \cdots.$$

Some member of this chain, $\langle x^{p^n} \rangle$ say, must be a serial subgroup of G , and it follows from Proposition 1 that $R(G)\langle x^{p^n} \rangle$ is also a serial subgroup of G . But

$R(G)\langle x^{p^n} \rangle = R(G)\langle x \rangle$, and we conclude that $A/R(G)$ is a subnormal subgroup of $G/R(G)$. If on the other hand $\langle x \rangle$ has finite order, and (R_i) is the infinite descending chain of G -subgroups of $R(G)$ whose existence is guaranteed by Lemma 8, then the chain $(R_i\langle x \rangle)$ is infinite, so that $R_i\langle x \rangle$ ser G for some i . Again we deduce from Proposition 1 that $A/R(G)$ is a subnormal subgroup of $G/R(G)$. This concludes the proof of Theorem C(ii).

4.3. For our proofs of B(viii) and B(ix), we need a lemma proved by Černikov in [10]. Because the proof is not difficult, we reproduce it here.

LEMMA 20. *If H is a finite subgroup of a group G and $(R_\alpha; \alpha \in A)$ is a descending chain of H -invariant subgroups of G , then $\bigcap (R_\alpha H; \alpha \in A) = (\bigcap (R_\alpha; \alpha \in A))H$.*

Proof. We need only show that $\bigcap (R_\alpha H) \leq (\bigcap R_\alpha)H$. If $g \in \bigcap (R_\alpha H)$, then we may write

$$g = r_\alpha h_\alpha \quad (r_\alpha \in R_\alpha, h_\alpha \in H)$$

for each α . If we set $A_h = \{\alpha; h_\alpha = h\}$ for each h , then obviously

$$g \in (\bigcap (R_\alpha; \alpha \in A_h))H \text{ for each } h.$$

However the subgroups $\bigcap (R_\alpha; \alpha \in A_h)H$ are finite in number and linearly ordered by inclusion, so that, because $A = \bigcup A_h$, we have

$$\bigcap (R_\alpha; \alpha \in A_h) = \bigcap (R_\alpha; \alpha \in A),$$

for some h . The Lemma follows.

We approach B(viii) through two preliminary results, each of type "if G satisfies the minimal condition on subgroups not having \mathcal{P} , and satisfies an additional condition, then either G is a Černikov group or all subgroups of G have \mathcal{P} ." We note that, in the first of these (in which the "additional condition" may be significantly weakened), we are not restricted to properties \mathcal{P} stronger than the property of either being a serial subgroup or being locally nilpotent.

PROPOSITION 5. *Let \mathfrak{X} be a subgroup-closed class of groups, and let G be a soluble locally finite group satisfying the minimal condition for non-normal non-locally- \mathfrak{X} subgroups. Either G is a Černikov group, or every subgroup of G is either a normal subgroup or a locally- \mathfrak{X} group.*

Proof. We suppose that G is neither a Černikov group nor a locally- \mathfrak{X} group, so that G has a finite subgroup $H \notin \mathfrak{X}$. Let

$$G = G_n > \cdots > G_1 = 1$$

be an invariant series each of whose factors is Abelian, and let r be the least

integer for which G_r is not a Černikov group. The subgroup A/G_{r-1} generated by all elements of G_r/G_{r-1} of prime order is an infinite Abelian residually finite normal subgroup, and HA/G_{r-1} is therefore also residually finite, so that there is a descending chain of non-trivial H -invariant subgroups of A/G_{r-1} with trivial intersection. If B/G_{r-1} is one of these subgroups, then BH is not locally- \mathfrak{X} ; thus, from Lemma 20 and the minimal condition satisfied by G , we have $HG_{r-1}/G_{r-1} \triangleleft G/G_{r-1}$. Because HG_{r-1} is a Černikov group, so is $G/C_G(HG_{r-1})$, from a result of Baer [3]. Therefore $C_G(HG_{r-1})$ is not a Černikov group, and there is an infinite residually finite Abelian subgroup of G centralizing H (cf. [20], p. 39). Another application of Lemma 20 yields that $H \triangleleft G$.

We have proved that each finite subgroup of G either is normal or is an \mathfrak{X} -group; it now follows very easily that each subgroup of G either is normal or is a locally- \mathfrak{X} group.

PROPOSITION 6. *Let \mathfrak{X} be a subgroup-closed class of locally nilpotent groups, and let G be a locally finite group satisfying the minimal condition for non-normal non-locally- \mathfrak{X} subgroups. Either G is a Černikov group, or G is soluble and every subgroup of G either is a normal subgroup or is a locally- \mathfrak{X} group.*

Proof. Again we may suppose that G is neither a Černikov group nor a locally- \mathfrak{X} group, so that G has a finite subgroup $H \notin \mathfrak{X}$; by Proposition 5, it suffices to prove that G is soluble.

Suppose that L is an intersection of an infinite descending chain of normal subgroups of G . From Lemma 20 and the minimal condition satisfied by G we have $LK \triangleleft G$ for each finite subgroup $K \geq H$; thus every finite subgroup of G/LH is a normal subgroup, and it follows easily that every subgroup is a normal subgroup. The group G/LH is therefore Abelian or Hamiltonian, and G/L is finite by Abelian. By Theorem A, G is locally soluble, so that G/L is soluble.

From this argument it follows first that the derived series of G breaks off after finitely many terms: if it did not, we could take L to be the intersection of these terms and deduce that G/L is soluble. So the limit P of the derived series of G is perfect and G/P is soluble. Appealing again to the above paragraph, we see that no proper subgroup of P can be an intersection of the terms of an infinite descending chain of G -subgroups of P . In other words, P satisfies the minimal condition for G -subgroups. However $P \leq R(G)$ by Theorem A, and every chief factor of G below P is therefore a central factor by Proposition 1. We conclude that P is a hypercentral group; and because P is also perfect it must be trivial (cf. Kuroš [21, p. 227]). Therefore G is soluble, and Proposition 6 follows.

4.4. We are now ready to give the proof of B(viii). We are given a subgroup-closed class \mathfrak{X} of locally nilpotent groups such that (a) \mathfrak{X} is closed under normal products of pairs of subgroups, or (b) \mathfrak{X} is quotient-closed and closed under

nilpotent subdirect products, or (c) \mathfrak{X} contains only locally finite groups; we must prove that if G is a \mathfrak{B} -group satisfying the minimal condition for non-normal non-locally- \mathfrak{X} subgroups, then either G is a Černikov group or every subgroup of G is normal or is a locally- \mathfrak{X} group.

If G is locally finite, this follows from Proposition 6. We therefore suppose that G is not locally finite (and therefore not a Černikov group). If G has a subgroup H which is neither normal nor a locally- \mathfrak{X} group, then there are elements $h \in H$ and $g \in G$ with $h^g \notin H$, and there is a finitely generated infinite subgroup $H_1 \leq H$ with $H_1 \notin \mathfrak{X}$; and we may replace G by $\langle g, h, H_1 \rangle$ and H by $H \cap \langle g, h, H_1 \rangle$ to assume that G is finitely generated and a counterexample to B(viii). Theorem A implies that G is finite by nilpotent and that $G/R(G) \in \mathcal{A}$. Let M be a subgroup minimal with respect to being neither normal nor an \mathfrak{X} -group. If M were finite, then, because G is residually finite, we could use Lemma 20 and the minimal condition satisfied by G to obtain a contradiction. Thus M is infinite and has an infinite cyclic factor group. It follows that M has a normal subgroup M_p of index p for each prime p . Because $M = M_p M_q$ for any distinct primes p and q , at most one of these subgroups can be normal in G . Thus M is a normal product of two infinite finitely generated \mathfrak{X} -groups, and a contradiction ensues if either \mathfrak{X} is closed under normal products or \mathfrak{X} contains only locally finite groups. We are left with the case in which \mathfrak{X} is quotient closed and closed under nilpotent subdirect products. Because M is a normal product of \mathfrak{X} -groups and because \mathfrak{X} -groups are locally nilpotent, M is certainly nilpotent, and so, because $M \notin \mathfrak{X}$, there will be a finite quotient group M/N of M with $M/N \notin \mathfrak{X}$. There is a prime p , coprime to $|M/N|$, for which M_p is an \mathfrak{X} -group, and for this p , we have $M = NM_p$. Thus M/N is isomorphic to $M_p/(M_p \cap N)$, which is an \mathfrak{X} -group. This contradiction completes the proof of B(viii).

4.5. The proof of assertion B(ix) is similar to (but easier than) the proof of B(viii), and again is preceded by a result concerning locally finite groups.

PROPOSITION 7. *If \mathfrak{X} is a subgroup-closed class of locally nilpotent groups, and if G is locally finite and satisfies the minimal condition for non-locally- \mathfrak{X} subgroups, then G is either a Černikov group or a locally- \mathfrak{X} group.*

Proof. By Theorem A, G is locally-nilpotent by finite; thus, if G is not a Černikov group, then it has an infinite descending chain (R_i) of normal subgroups, by Lemma 8. If G were also not a locally- \mathfrak{X} -group, then it would have a finite subgroup $H \notin \mathfrak{X}$, and the chain $(R_i X)$ would be an infinite descending chain of non-locally- \mathfrak{X} subgroups. Proposition 7 follows.

We now prove B(ix). Again, we have a subgroup-closed class \mathfrak{X} of locally nilpotent groups such that (a) \mathfrak{X} is closed under normal products of pairs of subgroups, or (b) \mathfrak{X} is quotient-closed and closed under nilpotent subdirect

products, or (c) \mathfrak{X} contains only locally finite groups; and we must prove that if G is a \mathfrak{B} -group satisfying the minimal condition for non-locally- \mathfrak{X} subgroups, then either G is a Černikov group or G is a locally- \mathfrak{X} group.

This follows from Proposition 7 if G is locally finite. We therefore suppose that G is not locally finite and must prove that G is a locally- \mathfrak{X} group. We suppose that this is not the case, and replace G by one of its infinite finitely generated subgroups which is not an \mathfrak{X} -group. By Theorem A, G is finite by nilpotent. Let M be a subgroup minimal with respect to not being an \mathfrak{X} -group; then M cannot be finite by Lemma 20 and the residual finiteness of G . Thus, M is infinite and has a normal \mathfrak{X} -subgroup M_p of index p for each prime p . A contradiction follows in just the same way as in the proof of B(viii), and this contradiction completes the proof of assertion B(ix).

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